# * Curve Fitting *

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If we have n points, $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

they can be written as

$$x: x_1 \quad x_2 \quad \dots \quad x_n ,$$
$$y: y_1 \quad y_2 \quad \dots \quad y_n$$

We need to get a function in a given form that can represent these points, $y_i \simeq y(x_i)$.

For example, if a quadratic form fn. is requried to fit these points, then we let $y = ax^2 + bx + c$ & it is to find a, b and c such that this fn best fits (represent) these point.

We find the constants a, b & c so as to minimize the error $S$, called the least square error,

$$S = \sum_{i=1}^{n} (y_i - y(x_i))^2$$

$$= (y_1 - y(x_1))^2 + (y_2 - y(x_2))^2 + \dots + (y_n - y(x_n))^2.$$

Constants a, b, c, .. are found in the least Squares sence by solving the Normal Equations.

## How to get the Normal Equations :-

1) Write $y(x)$ as a linear fn. in $a, b, c, \ldots$
2) Subsitute by each point in the Eq. form of $y(x)$. (But Don't Solve These Equations)
3) Get the Normal Equations as follow;
   - Multiply each equation of step ② by its Coff. of a, then add them $\Rightarrow$ 1st normal eq.
   - Multiply each equation of step ② by its Coff. of b, then add them $\Rightarrow$ 2nd normal eq.

   ⋮ ⋮ ⋮ ⋮ ⋮ ⋮

4) Solve the Normal Equations to get $a, b, c, \ldots$
5) If required the error, use $S = \sum_{i=1}^{n} \left(y_i - y(x_i)\right)^2$

We will prove the Normal Equations for the Case of a function of the form $y = ax^2 + bx + c$ that best fits some points $(x_i, y_i)$. We need to minimize the error $S = \sum_{i=1}^{n} (y_i - y(x_i))^2$

$$S = \sum_{i=1}^{n} (y_i - ax_i^2 - bx_i - c)^2$$

We solve the equations $\frac{\partial S}{\partial a} = 0$, $\frac{\partial S}{\partial b} = 0$ & $\frac{\partial S}{\partial c} = 0$

$$\frac{\partial S}{\partial a} = 0 \Rightarrow -2 \sum_i (y_i - ax_i^2 - b\,x_i - c)\,x_i^2 = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i x_i^2 = \sum_{i=1}^{n} (ax_i^2 + bx_i + c)\,x_i^2 \longrightarrow ①$$

نحصل على ① اذا ضربنا كل معادلة في معامل a ثم نجمع

$$\frac{\partial S}{\partial b} = 0 \Rightarrow -2 \sum_i (y_i - ax_i^2 - bx_i - c)\,x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} y_i x_i = \sum_{i=1}^{n} (ax_i^2 + bx_i + c)\,x_i \longrightarrow ②$$

نحصل على ② اذا ضربنا كل معادلة في معامل b ثم نجمع المعادلات

$$\frac{\partial S}{\partial c} = 0 \Rightarrow \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} (ax_i^2 + bx_i + c) \longrightarrow ③$$

نحصل على ③ نضرب معادلات النقاط $(x_i, y_i)$ في معامل c ثم نجمع.

Equations ①, ② & ③ are the normal Eq., we solve them to get a, b & c.

mple:- Find a quadratic function that best fits the points $(0,3)$ , $(1,1)$ , $(\frac{3}{2},\frac{7}{2})$ , $(-0.5,7.5)$

الحل :-

الداله المطلوبه داله تربيعيه فنفرضها على الصوره

$$y = ax^2 + bx + c$$

ونوجد الثوابت $a$ , $b$ , $c$ بحيث يكون الخطأ أقل

ما يمكن وذلك بحل (Normal Equations)

* أولاً نتأكد أن $y$ داله خطيه فى الثوابت المطلو

* نعوض بكل نقطه من النقط المعطاه فى الصوره

$$y = ax^2 + bx + c$$

فنحصل على ،

$$C = 3$$

$$a + b + C = 1$$

$$\frac{9}{4}a + \frac{3}{2}b + C = \frac{7}{2}$$

$$\frac{1}{4}a - \frac{1}{2}b + C = 7.5$$

هذه المعادلات نستخدمها للوصول الى ormal Equations

ولكن لا نقوم بحلهم مع بعض .

* نكون ( Normal Equations ) كالتالى :

- نضرب كل معادله فى معامل $a$ ثم نجمعهم فنحصل على

$$(c)+1(a+b+c)+\frac{9}{4}\left(\frac{9}{4}a+\frac{3}{2}b+c\right)$$

$$+\frac{1}{4}\left(\frac{1}{4}a-\frac{1}{2}b+c\right)=0(3)+1(1)+\frac{9}{4}\left(\frac{7}{2}\right)+\frac{1}{4}(7.5)$$

$$\Rightarrow \frac{49}{8}a+\frac{17}{4}b+\frac{7}{2}c=\frac{43}{4} \longrightarrow (1)$$

- نضرب كل معادلـه فى معامل $b$ ثم نجمعهم فنحصل على

$$(c)+1(a+b+c)+\frac{3}{2}\left(\frac{9}{4}a+\frac{3}{2}b+c\right)-\frac{1}{2}\left(\frac{1}{4}a-\frac{1}{2}b+c\right)$$

$$=0(3)+1(1)+\frac{3}{2}\left(\frac{7}{2}\right)-\frac{1}{2}(7.5)$$

$$\Rightarrow \frac{17}{4}a+\frac{7}{2}b+2c=\frac{5}{2} \longrightarrow (2)$$

- نضرب كل معادلـه فى معامل $c$ ثم نجمعهم فنحصل على

$$c)+1(a+b+c)+1\left(\frac{9}{4}a+\frac{3}{2}b+c\right)+1\left(\frac{1}{4}a-\frac{1}{2}b+c\right)$$

$$=1(3)+1(1)+1\left(\frac{7}{2}\right)+1(7.5)$$

$$\Rightarrow \frac{7}{2}a+2b+4c=15 \longrightarrow (3)$$

بحل هذه المعادلات نجد أن $=\frac{14}{3}$ , $b=\frac{-20}{3}$ & $c=3$

the best fitting quadratic function is ,

$$y(x)=\frac{14}{3}x^2-\frac{20}{3}x+3$$

~~Example:~~ What linear function best fits the data.

| $x$: | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $y$: | 0 | 1 | 1 | 2 |

Solution: The best fitting linear function is assummed as,

$$y = ax + b$$

Sub. by the given points, we get

$$a + b = 0$$
$$2a + b = 1$$
$$3a + b = 1$$
$$4a + b = 2$$

To get the normal equations, we multiply each of the above eqs. by its Coff. of a & then add them

$$(1 + 4 + 9 + 16)\,a + (1 + 2 + 3 + 4)\,b = 0 + 2 + 3 + 8$$

$$\Rightarrow 30a + 10b = 13 \longrightarrow (1)$$

Multiply the above Eqs. by their Coff. of b, then add

$$(1 + 2 + 3 + 4)\,a + (1 + 1 + 1 + 1)\,b = 0 + 1 + 1 + 2$$

$$\Rightarrow 10a + 4b = 4 \Rightarrow 5a + 2b = 2 \rightarrow (2)$$

Solving (1) & (2), we get $a = 3/5$, $b = -1/2$

the best fitting linear fn. is $y = \frac{3}{5}x - \frac{1}{2}$.

...ple:- Find an equation of the form $y = ax^2 + b$ that best fits the data

| | | | |
|---|---|---|---|
| x: | -1 | 0 | 1 |
| y: | 3.1 | 0.9 | 2.9 |

Solution: The best fitting fn. is $y = ax^2 + b$, sub. by these points, we get

$$a + b = 3.1$$
$$b = 0.9$$
$$a + b = 2.9$$

نضرب كل معادلة فى معامل الثابت $a$ ثم نجمع ،

$$(1+1)a + (1+1)b = 3.1 + 2.9$$

$$\Rightarrow a + b = 3 \rightarrow (1)$$

نضرب كل معادلة فى معامل الثابت $b$ ثم نجمع ،

$$(1+1)a + (1+1+1)b = 3.1 + 0.9 + 2.9$$

$$\Rightarrow 2a + 3b = 6.9 \longrightarrow (2)$$

بحل (1) ، (2) نوجد $a$ & $b$

$(2) - 2 \cdot (1) \Rightarrow b = 0.9$ & $a = 2.1$

The best fitting function is $y = 2.1x^2 + 0.9$

le: Find a function of the form $y = ae^{b/x}$, that best fits the data

| | | | | |
|---|---|---|---|---|
| $x$: | $-1$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $2$ |
| $y$: | $0.64$ | $0.32$ | $5.27$ | $1.84$ |

الحل: الشكل $y = ae^{b/x}$ ليس داله خطيه فى الثوابت $a, b$
ولذلك لا بد أن نأخذ $\ln$ للطرفين ،

$\ln y = \ln a + \frac{b}{x}$ => أصبحت خطيه => let $\begin{cases} \ln a = c \\ \ln y = z \end{cases}$

⇐ أصبحت شكل الداله $z = \frac{b}{x} + c$ حيث ،

| | | | | |
|---|---|---|---|---|
| $x$: | $-1$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | $2$ |
| $z = \ln y$: | $-0.44628$ | $-1.13943$ | $1.66203$ | $0.60976$ |

نعوض بكل نقطه فى العلاقه $z = \frac{b}{x} + c$ فنحصل على

$-b + c = -0.44628$ , $-2b + c = -1.13943$

$2b + c = 1.66203$ & $\frac{1}{2}b + c = 0.60976$

- نضرب فى معاملات $b$ ثم نجمع ⇐ $.25b - 0.5c = 6.35408$

- نضرب فى معاملات $c$ ثم نجمع ⇐ $0.5b + 4c = 0.68608$

بحل المعادلتين نجد أن $b = 0.701$ & $c = 0.26034$

$\Rightarrow \ln a = 0.26034 \Rightarrow a = 1.2973$

Best fitting fn. is $y = 1.2973\, e^{0.701/x}$

east Squares Error $= \sum (y_i - y(x_i))^2 = 0.00001864$